Procrastination makes you sick!

Researchers compared college students who were procrastinators and nonprocrastinators. Early in the semester, procrastinators reported fewer symptoms of illness, but late in the semester, they reported more symptoms than their nonprocrastinating peers.

In this section of the textbook, you will identify when both groups have the same number of symptoms as the point of intersection of two lines.

Objective #1: Decide whether an ordered pair is a solution of a linear system.

✔ *Solved Problem #1*

1. Determine if the ordered pair $(7,6)$ is a solution of the system: $\begin{cases} 2x-3y=-4 \\ 2x+y=4 \end{cases}$

To determine if $(7,6)$ is a solution to the system, replace x with 7 and y with 6 in both equations.

$$\begin{aligned} 2x-3y&=-4 \\ 2(7)-3(6)&=-4 \\ 14-18&=-4 \\ -4&=-4, \text{ true} \end{aligned} \qquad \begin{aligned} 2x+y&=4 \\ 2(7)+6&=4 \\ 14+6&=4 \\ 20&=4, \text{ false} \end{aligned}$$

The ordered pair does not satisfy both equations, so it is not a solution to the system.

Pencil Problem #1

1. Determine if the ordered pair $(2,3)$ is a solution of the system: $\begin{cases} x+3y=11 \\ x-5y=-13 \end{cases}$

Objective #2: Solve linear systems by substitution.

✔ *Solved Problem #2*

2. Solve by the substitution method: $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

Solve $2x+y=1$ for y.

$$\begin{aligned} 2x+y&=1 \\ y&=-2x+1 \end{aligned}$$

Pencil Problem #2

2. Solve by the substitution method: $\begin{cases} x+y=4 \\ y=3x \end{cases}$

Substitute: $3x + 2y = 4$

$$3x+2(\overbrace{-2x+1}^{y})=4$$
$$3x-4x+2=4$$
$$-x+2=4$$
$$-x=2$$
$$x=-2$$

Find y.

$$y=-2x+1$$
$$y=-2(-2)+1$$
$$y=5$$

The solution is $(-2,5)$.

The solution set is $\{(-2,5)\}$.

Objective #3: Solve linear systems by addition.

✔ *Solved Problem #3*

3. Solve the system: $\begin{cases}4x+5y=3\\2x-3y=7\end{cases}$

Multiply each term of the second equation by –2 and add the equations to eliminate *x*.

$$\begin{aligned}4x+5y&=3\\ \underline{-4x+6y}&\underline{=-14}\\ 11y&=-11\\ y&=-1\end{aligned}$$

Back-substitute into either of the original equations to solve for x.

$$2x-3y=7$$
$$2x-3(-1)=7$$
$$2x+3=7$$
$$2x=4$$
$$x=2$$

The solution set is $\{(2,-1)\}$.

Pencil Problem #3

3. Solve the system: $\begin{cases}3x-4y=11\\2x+3y=-4\end{cases}$

Objective #4: Identify systems that do not have exactly one ordered-pair solution.

✔ *Solved Problem #4*

4a. Solve the system: $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Multiply the first equation by 2, and then add the equations.

$$\begin{array}{r} 10x-4y=8 \\ \underline{-10x+4y=7} \\ 0=15 \end{array}$$

Since there are no pairs (x, y) for which 0 will equal 15, the system is inconsistent and has no solution.
The solution set is $\varnothing$ or $\{\ \ \}$.

4b. Solve the system: $\begin{cases} x=4y-8 \\ 5x-20y=-40 \end{cases}$

Substitute $4y-8$ for x in the second equation.

$$\begin{aligned} 5x-20y&=-40 \\ 5(\overbrace{4y-8}^{x})-20y&=-40 \\ 20y-40-20y&=-40 \\ -40&=-40 \end{aligned}$$

Since $-40=-40$ for all values of *x* and *y*, the system is dependent.

The solution set is $\{(x, y)|x=4y-8\}$ or $\{(x, y)|5x-20y=-40\}$.

Pencil Problem #4

4a. Solve the system: $\begin{cases} x=9-2y \\ x+2y=13 \end{cases}$

4b. Solve the system: $\begin{cases} y=3x-5 \\ 21x-35=7y \end{cases}$

Objective #5: Solve problems using systems of linear equations.

✔ *Solved Problem #5*

5. A company that manufactures running shoes has a fixed cost of \$300,000. Additionally, it costs \$30 to produce each pair of shoes. The shoes are sold at \$80 per pair.

5a. Write the cost function, *C*, of producing *x* pairs of running shoes.

$$C(x)=\overbrace{300,000}^{\text{fixed costs}}+\overbrace{30x}^{\$30\text{ per pair}}$$

Pencil Problem #5

5. A company that manufactures small canoes has a fixed cost of \$36,000. Additionally, it costs \$40 to produce each canoe. The selling price is \$160 per canoe.

5a. Write the cost function, *C*, of producing *x* canoes.

5b. Write the revenue function, R, from the sale of x pairs of running shoes.

$$R(x) = \overbrace{80x}^{\$80\text{ per pair}}$$

5c. Determine the break-even point. Describe what this means.

The system is $\begin{cases} y = 300,000 + 30x \\ y = 80x \end{cases}$

The break-even point is where $R(x) = C(x)$.

$$\begin{aligned} R(x) &= C(x) \\ 80x &= 300,000 + 30x \\ 50x &= 300,000 \\ x &= 6000 \end{aligned}$$

Back-substitute to find y: $\begin{aligned} y &= 80x \\ y &= 80(6000) \\ y &= 480,000 \end{aligned}$

The break-even point is (6000, 480,000).

This means the company will break even when it produces and sells 6000 pairs of shoes. At this level, both revenue and costs are \$480,000.

5b. Write the revenue function, R, from the sale of x canoes.

5c. Determine the break-even point. Describe what this means.

<u>**Answers** for Pencil Problems *(Textbook Exercise references in parentheses)*</u>:

1. The ordered pair is a solution to the system. *(5.1 #1)*

2. $\{(1,3)\}$ *(5.1 #5)*

3. $\{(1,-2)\}$ *(5.1 #27)*

4a. $\varnothing$ or $\{\ \ \}$ *(5.1 #31)*

4b. $\{(x,y) \mid y = 3x - 5\}$ or $\{(x,y) \mid 21x - 35 = 7y\}$ *(5.1 #33)*

5a. $C(x) = 36,000 + 40x$ *(5.1 #61a)* **5b.** $R(x) = 160x$ *(5.1 #61b)*

5c. Break-even point: (300, 48,000). Which means when 300 canoes are produced the company will break-even with cost and revenue at \$48,000. *(5.1 #61c)*

Section 5.2
Systems of Linear Equations in Three Variables

Hit the BRAKES!

Did you know that a mathematical model can be used to describe the relationship between the number of feet a car travels once the brakes are applied and the number of seconds the car is in motion after the brakes are applied?

In the Exercise Set of this section, using data collected by a research firm, you will be asked to write the mathematical model that describes this situation.

Objective #1: Verify the solution of a system of linear equations in three variables.

✔ *Solved Problem #1*

1. Show that the ordered triple $(-1,-4,5)$ is a solution of the system:

$$\begin{cases} x-2y+3z=22 \\ 2x-3y-z=5 \\ 3x+y-5z=-32 \end{cases}$$

Test the ordered triple in each equation.

$$\begin{aligned} x-2y+3z &= 22 \\ (-1)-2(-4)+3(5) &= 22 \\ 22 &= 22, \text{ true} \end{aligned}$$

$$\begin{aligned} 2x-3y-z &= 5 \\ 2(-1)-3(-4)-(5) &= 5 \\ 5 &= 5, \text{ true} \end{aligned}$$

$$\begin{aligned} 3x+y-5z &= -32 \\ 3(-1)+(-4)-5(5) &= -32 \\ -32 &= -32, \text{ true} \end{aligned}$$

The ordered triple (–1, –4, 5) makes all three equations true, so it is a solution of the system.

Pencil Problem #1

1. Determine if the ordered triple $(2,-1,3)$ is a solution of the system:

$$\begin{cases} x+y+z=4 \\ x-2y-z=1 \\ 2x-y-z=-1 \end{cases}$$

Objective #2: Solve systems of linear equations in three variables.

✔ Solved Problem #2

2. Solve the system: $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

Add the first two equations to eliminate *z*.

$$\begin{array}{l} x+4y-z=20 \\ \underline{3x+2y+z=\ \ 8} \\ 4x+6y\qquad =28 \end{array}$$

Multiply the first equation by 2 and add it to the third equation to eliminate *z* again.

$$\begin{array}{l} 2x+8y-2z=40 \\ \underline{2x-3y+2z=-16} \\ 4x+5y\qquad =24 \end{array}$$

Solve the system of two equations in two variables.

$$4x+6y=28$$
$$4x+5y=24$$

Multiply the second equation by –1 and add the equations.

$$\begin{array}{r} 4x+6y=28 \\ \underline{-4x-5y=-24} \\ y=4 \end{array}$$

Back-substitute 4 for *y* to find *x*.

$$\begin{aligned} 4x+6y&=28 \\ 4x+6(4)&=28 \\ 4x+24&=28 \\ 4x&=4 \\ x&=1 \end{aligned}$$

Back-substitute into an original equation.

$$\begin{aligned} 3x+2y+z&=8 \\ 3(1)+2(4)+z&=8 \\ 11+z&=8 \\ z&=-3 \end{aligned}$$

The solution is $(1,4,-3)$ and the solution set is $\{(1,4,-3)\}$.

Pencil Problem #2

2. Solve the system: $\begin{cases} 4x-y+2z=11 \\ x+2y-z=-1 \\ 2x+2y-3z=-1 \end{cases}$

Objective #3: Solve problems using systems in three variables.

✔ *Solved Problem #3*

3. Find the quadratic function $y = ax^2 + bx + c$ whose graph passes through the points $(1,4)$, $(2,1)$, and $(3,4)$.

Use each ordered pair to write an equation.

$(1,4)$:
$$\begin{aligned} y &= ax^2 + bx + c \\ 4 &= a(1)^2 + b(1) + c \\ 4 &= a + b + c \end{aligned}$$

$(2,1)$:
$$\begin{aligned} y &= ax^2 + bx + c \\ 1 &= a(2)^2 + b(2) + c \\ 1 &= 4a + 2b + c \end{aligned}$$

$(3,4)$:
$$\begin{aligned} y &= ax^2 + bx + c \\ 4 &= a(3)^2 + b(3) + c \\ 4 &= 9a + 3b + c \end{aligned}$$

The system of three equations in three variables is:

$$\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 1 \\ 9a + 3b + c = 4 \end{cases}$$

Solve the system: $\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 1 \\ 9a + 3b + c = 4 \end{cases}$

Multiply the first equation by –1 and add it to the second equation:

$$\begin{array}{r} -a - b - c = -4 \\ 4a + 2b + c = 1 \\ \hline 3a + b = -3 \end{array}$$

(continued on next page)

Pencil Problem #3

3. Find the quadratic function $y = ax^2 + bx + c$ whose graph passes through the points $(-1,6)$, $(1,4)$, and $(2,9)$.

Multiply the first equation by –1 and add it to the third equation:

$$\begin{array}{l} -a-b-c=-4 \\ \underline{9a+3b+c=\ \ 4} \\ 8a+2b \quad\ \ = \ \ 0 \end{array}$$

Solve this system of two equations in two variables.

$$\begin{aligned} 3a+b&=-3 \\ 8a+2b&=0 \end{aligned}$$

Multiply the first equation by –2 and add to the second equation: $-6a-2b=6$

$$\begin{aligned} \underline{8a+2b=0} \\ 2a=6 \\ a=3 \end{aligned}$$

Back-substitute to find b: $3a+b=-3$

$$\begin{aligned} 3(3)+b&=-3 \\ 9+b&=-3 \\ b&=-12 \end{aligned}$$

Back-substitute into an original equation to find c:

$$\begin{aligned} a+b+c&=4 \\ (3)+(-12)+c&=4 \\ -9+c&=4 \\ c&=13 \end{aligned}$$

The quadratic function is $y=3x^2-12x+13$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. not a solution *(5.2 #1)*

2. $\{(2,-1,1)\}$ *(5.2 #7)*

3. $y=2x^2-x+3$ *(5.2 #19)*

Section 5.3
Partial Fractions

Where's the "UNDO" Button?

We have learned how to write a sum or difference of rational expressions as a single rational expression and saw how this skill is necessary when solving rational inequalities. However, in calculus, it is sometimes necessary to write a single rational expression as a sum or difference of simpler rational expressions, undoing the process of adding or subtracting.
In this section, you will learn how to break up a rational expression into sums or differences of rational expressions with simpler denominators.

Objective #1: Decompose $\frac{P}{Q}$, where Q has only distinct linear factors.

✔ *Solved Problem #1*

1. Find the partial fraction decomposition of $\frac{5x-1}{(x-3)(x+4)}$.

Write a constant over each distinct linear factor in the denominator.

$$\frac{5x-1}{(x-3)(x+4)}=\frac{A}{x-3}+\frac{B}{x+4}$$

Multiply by the LCD, $(x-3)(x+4)$, to eliminate fractions. Then simplify and rearrange terms.

$$(x-3)(x+4)\frac{5x-1}{(x-3)(x+4)}=(x-3)(x+4)\frac{A}{x-3}+(x-3)(x+4)\frac{B}{x+4}$$
$$5x-1=A(x+4)+B(x-3)$$
$$5x-1=Ax+4A+Bx-3B$$
$$5x-1=(A+B)x+(4A-3B)$$

Equating the coefficients of x and equating the constant terms, we obtain a system of equations.

$$\begin{cases} A+B=5 \\ 4A-3B=-1 \end{cases}$$

Multiplying the first equation by 3 and adding it to the second equation, we obtain $7A = 14$, so $A = 2$. Substituting 2 for A in either equation, we obtain $B = 3$. The partial fraction decomposition is

$$\frac{5x-1}{(x-3)(x+4)}=\frac{2}{x-3}+\frac{3}{x+4}.$$

Pencil Problem #1

1. Find the partial fraction decomposition of $\dfrac{3x+50}{(x-9)(x+2)}$.

Objective #2: Decompose $\dfrac{P}{Q}$, where Q has repeated linear factors.

✔ Solved Problem #2

2. Find the partial fraction decomposition of $\dfrac{x+2}{x(x-1)^2}$.

Include one fraction for each power of $x - 1$.

$$\frac{x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiply by the LCD, $x(x-1)^2$, to eliminate fractions. Then simplify and rearrange terms.

$$x(x-1)^2 \frac{x+2}{x(x-1)^2} = x(x-1)^2 \frac{A}{x} + x(x-1)^2 \frac{B}{x-1} + x(x-1)^2 \frac{C}{(x-1)^2}$$

$$x+2 = A(x-1)^2 + Bx(x-1) + Cx$$

$$x+2 = A(x^2-2x+1) + Bx(x-1) + Cx$$

$$x+2 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$

$$0x^2 + x + 2 = (A+B)x^2 + (-2A - B + C)x + A$$

Equating the coefficients of like terms, we obtain a system of equations.

$$\begin{cases} A + B = 0 \\ -2A - B + C = 1 \\ A = 2 \end{cases}$$

We see immediately that $A = 2$. Substituting 2 for A in the first equation, we obtain $B = -2$. Substituting these values into the second equation, we obtain $C = 3$. The partial fraction decomposition is

$$\frac{x+2}{x(x-1)^2} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2} \text{ or } \frac{2}{x} - \frac{2}{x-1} + \frac{3}{(x-1)^2}.$$

Pencil Problem #2

2. Find the partial fraction decomposition of $\dfrac{x^2}{(x-1)^2(x+1)}$.

Objective #3: Decompose $\frac{P}{Q}$, where Q has a nonrepeated prime quadratic factor.

✔ ***Solved Problem #3***

3. Find the partial fraction decomposition of $\dfrac{8x^2+12x-20}{(x+3)(x^2+x+2)}$.

Use a constant over the linear factor and a linear expression over the prime quadratic factor.

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+x+2}$$

Multiply by the LCD, $(x+3)(x^2+x+2)$, to eliminate fractions. Then simplify and rearrange terms.

$$(x+3)(x^2+x+2)\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=(x+3)(x^2+x+2)\frac{A}{x+3}+(x+3)(x^2+x+2)\frac{Bx+C}{x^2+x+2}$$

$$8x^2+12x-20=A(x^2+x+2)+(Bx+C)(x+3)$$

$$8x^2+12x-20=Ax^2+Ax+2A+Bx^2+3Bx+Cx+3C$$

$$8x^2+12x-20=(A+B)x^2+(A+3B+C)x+(2A+3C)$$

Equating the coefficients of like terms, we obtain a system of equations.

$$\begin{cases} A+B=8 \\ A+3B+C=12 \\ 2A+3C=-20 \end{cases}$$

Multiply the second equation by –3 and add to the third equation to obtain $-A-9B=-56$. Add this result to the first equation in the system above to obtain $-8B=-48$, so $B=6$. Substituting this value into the first equation, we obtain $A=2$. Substituting the value of A into the third equation, we obtain $C=-8$.

The partial fraction decomposition is

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{2}{x+3}+\frac{6x-8}{x^2+x+2}.$$

✎ ***Pencil Problem #3*** ✎

3. Find the partial fraction decomposition of $\dfrac{5x^2+6x+3}{(x+1)(x^2+2x+2)}$.

Objective #4: $\frac{P}{Q}$, where Q has a prime, repeated quadratic factor

✔ ***Solved Problem #4***

4. Find the partial fraction decomposition of $\frac{2x^3+x+3}{(x^2+1)^2}$.

Include one fraction with a linear numerator for each power of x^2+1.

$$\frac{2x^3+x+3}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$$

Multiply by the LCD, $(x^2+1)^2$, to eliminate fractions. Then simplify and rearrange terms.

$$(x^2+1)^2\frac{2x^3+x+3}{(x^2+1)^2}=(x^2+1)^2\frac{Ax+B}{x^2+1}+(x^2+1)^2\frac{Cx+D}{(x^2+1)^2}$$

$$2x^3+x+3=(Ax+B)(x^2+1)+Cx+D$$

$$2x^3+x+3=Ax^3+Ax+Bx^2+B+Cx+D$$

$$2x^3+x+3=Ax^3+Bx^2+(A+C)x+(B+D)$$

Equating the coefficients of like terms, we obtain a system of equations.

$$\begin{cases} A=2 \\ B=0 \\ A+C=1 \\ B+D=3 \end{cases}$$

We immediately see that $A = 2$ and $B = 0$. By performing appropriate substitutions, we obtain $C = -1$ and $D = 3$. The partial fraction decomposition is

$$\frac{2x^3+x+3}{(x^2+1)^2}=\frac{2x}{x^2+1}+\frac{-x+3}{(x^2+1)^2}.$$

Pencil Problem #4

4. Find the partial fraction decomposition of $\dfrac{x^3+x^2+2}{(x^2+2)^2}$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $\dfrac{7}{x-9}-\dfrac{4}{x+2}$ (5.3 #11)

2. $\dfrac{1}{4(x+1)}+\dfrac{3}{4(x-1)}+\dfrac{1}{2(x-1)^2}$ (5.3 #27)

3. $\dfrac{2}{x+1}+\dfrac{3x-1}{x^2+2x+2}$ (5.3 #31)

4. $\dfrac{x+1}{x^2+2}-\dfrac{2x}{(x^2+2)^2}$ (5.3 #37)

Section 5.4
Systems of Nonlinear Equations in Two Variables

DO YOU FEEL SAFE????

Scientists debate the probability that a "doomsday rock" will collide with Earth. It has been estimated that an asteroid crashes into Earth about once every 250,000 years, and that such a collision would have disastrous results.

Understanding the path of Earth and the path of a comet is essential to detecting threatening space debris. Orbits about the sun are not described by linear equations. The ability to solve systems that do not contain linear equations provides NASA scientists watching for troublesome asteroids with a way to locate possible collision points with Earth's orbit.

Objective #1: Recognize systems of nonlinear equations in two variables.

✔ *Solved Problem #1*

1a. True or false: A solution of a nonlinear system in two variables is an ordered pair of real numbers that satisfies at least one equation in the system.

False; a solution must satisfy *all* equations in the system.

1b. True or false: The solution of a system of nonlinear equations corresponds to the intersection points of the graphs in the system.

True; each solution will correspond to an intersection point of the graphs.

Pencil Problem #1

1a. True or false: A system of nonlinear equations cannot contain a linear equation.

1b. True or false: The graphs of the equations in a nonlinear system could be a parabola and a circle.

Objective #2: Solve nonlinear systems by substitution.

✔ *Solved Problem #2*

2. Solve by the substitution method:

$$\begin{cases} x+2y=0 \\ (x-1)^2+(y-1)^2=5 \end{cases}$$

Solve the first equation for *x*: $x+2y=0$

$$x=-2y$$

Substitute the expression $-2y$ for *x* in the second equation and solve for *y*.

$$(x-1)^2+(y-1)^2=5$$

$$(\overbrace{-2y}^{x}-1)^2+(y-1)^2=5$$

$$4y^2+4y+1+y^2-2y+1=5$$

$$5y^2+2y-3=0$$

$$(5y-3)(y+1)=0$$

$5y-3=0$ or $y+1=0$

$y=\frac{3}{5}$ or $y=-1$

If $y=\frac{3}{5}$, $x=-2\left(\frac{3}{5}\right)=-\frac{6}{5}$.

If $y=-1$, $x=-2(-1)=2$.

Check $(2,-1)$ in both original equations.

$x+2y=0$

$2+2(-1)=0$

$0=0$, true

$(x-1)^2+(y-1)^2=5$

$(2-1)^2+(-1-1)^2=5$

$1+4=5$

$5=5$, true

Check $\left(-\frac{6}{5},\frac{3}{5}\right)$ in both original equations.

$x+2y=0$

$-\frac{6}{5}+2\left(\frac{3}{5}\right)=0$

$-\frac{6}{5}+\frac{6}{5}=0$

$0=0$, true

$(x-1)^2+(y-1)^2=5$

$\left(-\frac{6}{5}-1\right)^2+\left(\frac{3}{5}-1\right)^2=5$

$\frac{121}{25}+\frac{4}{25}=5$

$\frac{125}{25}=5$

$5=5$, true

The solution set is $\left\{\left(-\frac{6}{5},\frac{3}{5}\right),(2,-1)\right\}$.

Pencil Problem #2

2. Solve by the substitution method:

$$\begin{cases} x+y=2 \\ y=x^2-4x+4 \end{cases}$$

Objective #3: Solve nonlinear systems by addition.

✔ *Solved Problem #3*

3. Solve by the addition method:

$$\begin{cases} y = x^2 + 5 \\ x^2 + y^2 = 25 \end{cases}$$

Arrange the first equation so that variable terms appear on the left, and constants appear on the right.
Add the resulting equations to eliminate the x^2-terms and solve for *y*.

$$\begin{array}{r} -x^2 + y = 5 \\ \underline{x^2 + y^2 = 25} \\ y^2 + y = 30 \end{array}$$

Solve the resulting quadratic equation.

$$y^2 + y - 30 = 0$$
$$(y+6)(y-5) = 0$$
$$y+6=0 \quad \text{or} \quad y-5=0$$
$$y=-6 \quad \text{or} \quad y=5$$

If $y=-6$,

$$x^2 + (-6)^2 = 25$$
$$x^2 + 36 = 25$$
$$x^2 = -11$$

When $y=-6$ there is no real solution.

If $y=5$,

$$x^2 + (5)^2 = 25$$
$$x^2 + 25 = 25$$
$$x^2 = 0$$
$$x = 0$$

Check $(0,5)$ in both original equations.

$y = x^2 + 5$	$x^2 + y^2 = 25$
$5 = (0)^2 + 5$	$0^2 + 5^2 = 25$
$5 = 5$, true	$25 = 25$, true

The solution set is $\{(0,5)\}$.

Pencil Problem #3

3. Solve by the addition method:

$$\begin{cases} x^2 + y^2 = 13 \\ x^2 - y^2 = 5 \end{cases}$$

Objective #4: Solve problems using systems of nonlinear equations.

✔ Solved Problem #4

4. Find the length and width of a rectangle whose perimeter is 20 feet and whose area is 21 square feet.

The system is $\begin{cases} 2x+2y=20 \\ xy=21. \end{cases}$

Solve the second equation for x: $xy=21$

$$x=\frac{21}{y}$$

Substitute the expression $\frac{21}{y}$ for x in the first equation and solve for y.

$$\begin{aligned} 2x+2y&=20 \\ 2\left(\frac{21}{y}\right)+2y&=20 \\ \frac{42}{y}+2y&=20 \\ 42+2y^2&=20y \\ 2y^2-20y+42&=0 \\ y^2-10y+21&=0 \\ (y-7)(y-3)&=0 \\ y-7=0 \quad \text{or} \quad y-3&=0 \\ y=7 \quad \text{or} \quad y&=3 \end{aligned}$$

If $y=7$, $x=\frac{21}{7}=3$.

If $y=3$, $x=\frac{21}{3}=7$.

The dimensions are 7 feet by 3 feet.

Pencil Problem #4

4. The sum of two numbers is 10 and their product is 24. Find the numbers.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. false *(5.4 #1)* **1b.** true *(5.4 #27)* **2.** $\{(1,1),(2,0)\}$ *(5.4 #3)*

3. $\{(-3,-2),(-3,2),(3,-2),(3,2)\}$ *(5.4 #19)* **4.** 6 and 4 *(5.4 #43)*

Section 5.5
Systems of Inequalities

The Beat Goes On!

Normal heart rates for children vary by age. However, both the ages, in years, and the heart rates, in beats per minute, are given in ranges rather than as specific values.

Age (years)	**Awake Rate** (beats per minute)	**Sleeping Rate** (beats per minute)
3-5	65-110	65-100
6-11	60-95	58-90
12-15	55-85	50-90

This section of the textbook will demonstrate how systems of linear inequalities in two variables can enable you to establish healthy heart rate ranges for children age 3 to 15.

Objective #1: Graph a linear inequality in two variables.

✔ *Solved Problem #1*

1a. Graph: $4x - 2y \geq 8$.

First, graph the equation $4x - 2y = 8$ with a solid line.

Find the x–intercept:

$$\begin{aligned} 4x - 2y &= 8 \\ 4x - 2(0) &= 8 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

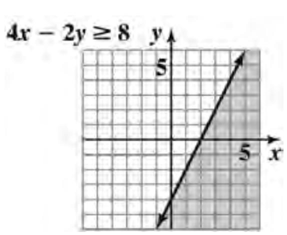
Find the y–intercept:

$$\begin{aligned} 4x - 2y &= 8 \\ 4(0) - 2y &= 8 \\ -2y &= 8 \\ y &= -4 \end{aligned}$$

Next, use the origin as a test point.

$$\begin{aligned} 4x - 2y &\geq 8 \\ 4(0) - 2(0) &\geq 8 \\ 0 &\geq 8, \text{ false} \end{aligned}$$

Since the statement is false, shade the half-plane that does not contain the test point



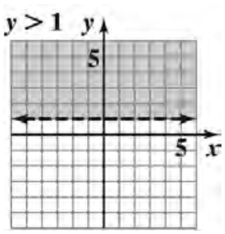
Pencil Problem #1

1a. Graph: $x - 2y > 10$.

1b. Graph: $y > 1$.

Graph the line $y = 1$ with a dashed line.

Since the inequality is of the form $y > a$, shade the half-plane above the line.



1b. Graph: $x \le 1$.

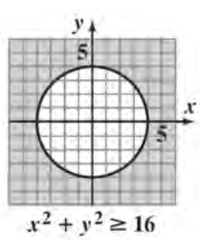
Objective #2: Graph a nonlinear inequality in two variables..

✔ Solved Problem #2

2. Graph: $x^2 + y^2 \ge 16$.

The graph of $x^2 + y^2 = 16$ is a circle of radius 4 centered at the origin. Use a solid circle because equality is included in ≥.

The point (0, 0) is not on the circle, so we use it as a test point. The result is 0 ≥ 16, which is false. Since the point (0, 0) is inside the circle, the region outside the circle belongs to the solution set. Shade the region outside the circle.



Pencil Problem #2

2. Graph: $x^2 + y^2 > 25$.

Objective #3: Use mathematical models involving systems of linear inequalities.

✔ Solved Problem #3

3. The healthy weight region for men and women ages 19 to 34 can be modeled by the following system of linear inequalities:

$$\begin{cases} x+y \geq 70 \\ 2.6x+y \leq 115 \end{cases}$$

Show that (6, 90) is a solution of the system of inequalities that describes healthy weight for this age group.

Substitute the coordinates of (6, 90) into both inequalities of the system.

$$\begin{cases} x+y \geq 70 \\ 2.6x+y \leq 115 \end{cases}$$

$$\begin{aligned} x+y &\geq 70 \\ 6+90 &\geq 70, \text{ true} \end{aligned}$$

$$\begin{aligned} 2.6x+y &\leq 115 \\ 2.6(6)+90 &\leq 115 \\ 105.6 &\leq 115, \text{ true} \end{aligned}$$

(6, 90) is a solution of the system.

Pencil Problem #3

3. The normal respiratory rate region for children ages 3 to 15 can be modeled by the following system of linear inequalities:

$$\begin{cases} 0.82x+y \geq 24 \\ 0.76x+y \leq 27 \end{cases}$$

Show that (5, 22) is a solution of the system of inequalities that describes normal respiratory rate for this age group.

Objective #4: Graph a system of linear inequalities.

✔ Solved Problem #4

4. Graph the solution set of the system:

$$\begin{cases} x-3y<6 \\ 2x+3y \geq -6 \end{cases}$$

Graph the line $x-3y=6$ with a dashed line.
Graph the line $2x+3y=-6$ with a solid line.

For $x-3y<6$ use a test point such as (0, 0).

$$\begin{aligned} x-3y &< 6 \\ 0-3(0) &< 6 \\ 0 &< 6, \text{ true} \end{aligned}$$

Since the statement is true, shade the half-plane that contains the test point.

Pencil Problem #4

4. Graph the solution set of the system:

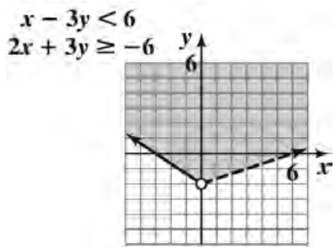
$$\begin{cases} y>2x-3 \\ y<-x+6 \end{cases}$$

For $2x+3y \ge -6$ use a test point such as (0, 0).

$$\begin{aligned} 2x+3y &\ge -6 \\ 2(0)+3(0) &\ge -6 \\ 0 &\ge -6, \text{ true} \end{aligned}$$

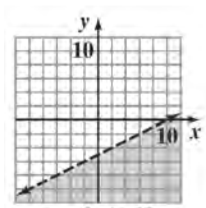
Since the statement is true, shade the half-plane that contains the test point.

The solution set of the system is the intersection (the overlap) of the two half-planes.



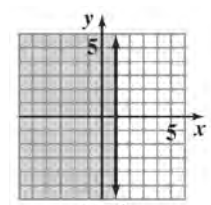
Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a.



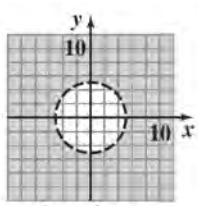
$x - 2y > 10$ *(5.5 #3)*

1b.



$x \le 1$ *(5.5 #9)*

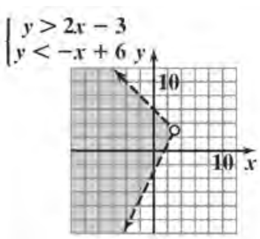
2.



$x^2 + y^2 > 25$ *(5.5 #15)*

3. $0.82x + y \ge 24$; $0.82(5) + 22 \ge 24$; $26.1 \ge 24$, true $\qquad$ $0.76x + y \le 27$; $0.76(5) + 22 \le 27$; $25.8 \le 27$, true *(5.5 #77)*

4.



(5.5 #31)

Section 5.6
Linear Programming

MAXIMUM Output with *MINIMUM* Effort!

Many situations in life involve quantities that must be maximized or minimized. Businesses are interested in maximizing profit and minimizing costs.

In the Exercise Set for this section of the textbook, you will encounter a manufacturer looking to maximize profits from selling two models of mountain bikes. But there is a limited amount of time available to assemble and paint these bikes. We will learn how to balance these constraints and determine the proper number of each bike that should be produced.

Objective #1:
Write an objective function describing a quantity that must be maximized or minimized.

✔ *Solved Problem #1*

1. A company manufactures bookshelves and desks for computers. Let *x* represent the number of bookshelves manufactured daily and *y* the number of desks manufactured daily. The company's profits are $25 per bookshelf and $55 per desk.

Write the objective function that models the company's total daily profit, *z*, from *x* bookshelves and *y* desks.

The total profit is 25 times the number of bookshelves, *x*, plus 55 times the number of desks, *y*.

$$\overbrace{z}^{\text{profit}} = \overbrace{25x}^{\substack{\$25\text{ for each}\\\text{bookshelf}}} + \overbrace{55y}^{\substack{\$55\text{ for each}\\\text{desk}}}$$

The objective function is $z = 25x + 55y$.

Pencil Problem #1

1. A television manufacturer makes QLED and OLED televisions. The profit per unit is $125 for the QLED televisions and $200 for the OLED televisions. Let x = the number of QLED televisions manufactured in a month and y = the number of OLED televisions manufactured in a month.

Write the objective function that models the total monthly profit.

Objective #2: Use inequalities to describe limitations in a situation.

✔ Solved Problem #2

2. Recall that the company in *Solved Problem #1* manufactures bookshelves and desks for computers. x represents the number of bookshelves manufactured daily and y the number of desks manufactured daily.

2a. Write an inequality that models the following constraint: To maintain high quality, the company should not manufacture more than a total of 80 bookshelves and desks per day.

$x + y \le 80$

2b. Write an inequality that models the following constraint: To meet customer demand, the company must manufacture between 30 and 80 bookshelves per day, inclusive.

$30 \le x \le 80$

2c. Write an inequality that models the following constraint: The company must manufacture at least 10 and no more than 30 desks per day.

$10 \le y \le 30$

2d. Summarize what you have described about this company by writing the objective function for its profits (from *Solved Problem #1*) and the three constraints.

Objective function: $z = 25x + 55y$.

Constraints: $\begin{cases} x + y \le 80 \\ 30 \le x \le 80 \\ 10 \le y \le 30 \end{cases}$

Pencil Problem #2

2. Recall that the manufacturer in *Pencil Problem #1* makes QLED and OLED televisions. x represents the number of QLED televisions manufactured monthly and y the number of OLED televisions manufactured monthly.

2a. Write an inequality that models the following constraint: Equipment in the factory allows for making at most 450 QLED televisions in one month.

2b. Write an inequality that models the following constraint: Equipment in the factory allows for making at most 200 OLED televisions in one month.

2c. Write an inequality that models the following constraint: The cost to the manufacturer per unit is \$600 for the QLED televisions and \$900 for the OLED televisions. Total monthly costs cannot exceed \$360,000.

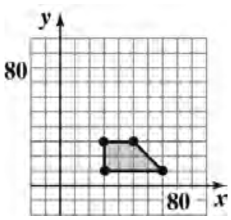
2d. Summarize what you have described about this company by writing the objective function for its profits (from *Pencil Problem #1*) and the three constraints.

Objective #3: Use linear programming to solve problems.

✔ *Solved Problem #3*

3a. For the company in *Solved Problems #1 and 2*, how many bookshelves and how many desks should be manufactured per day to obtain maximum profit? What is the maximum daily profit?

Graph the constraints and find the corners, or vertices, of the region of intersection.



Find the value of the objective function at each corner of the graphed region.

Corner (x, y)	Objective Function $z = 25x + 55y$
$(30,10)$	$z = 25(30)+55(10)$ $= 750+550$ $=1300$
$(30,30)$	$z = 25(30)+55(30)$ $= 750+1650$ $= 2400$
$(50,30)$	$z = 25(50)+55(30)$ $=1250+1650$ $= 2900$ (Maximum)
$(70,10)$	$z = 25(70)+55(10)$ $=1750+550$ $= 2300$

The maximum value of z is 2900 and it occurs at the point (50, 30).

In order to maximize profit, 50 bookshelves and 30 desks must be produced each day for a profit of \$2900.

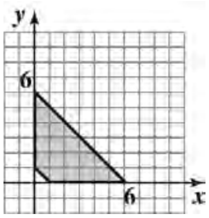
Pencil Problem #3

3a. For the company in *Pencil Problems #1 and 2*, how many QLED and OLED televisions should be manufactured per month to obtain maximum profit? What is the maximum monthly profit?

3b. Find the maximum value of the objective function $z = 3x + 5y$ subject to the constraints:

$$\begin{cases} x \geq 0, \quad y \geq 0 \\ x + y \geq 1 \\ x + y \leq 6 \end{cases}$$

Graph the region that represents the intersection of the constraints:



Find the value of the objective function at each corner of the graphed region.

Corner (x, y)	Objective Function $z = 3x + 5y$
$(0,1)$	$z = 3(0) + 5(1) = 5$
$(1,0)$	$z = 3(1) + 5(0) = 3$
$(0,6)$	$z = 3(0) + 5(6) = 30$ (Maximum)
$(6,0)$	$z = 3(6) + 5(0) = 18$

The maximum value is 30.

3b. Find the maximum value of the objective function $z = 4x + y$ subject to the constraints:

$$\begin{cases} x \geq 0, \quad y \geq 0 \\ 2x + 3y \leq 12 \\ x + y \geq 3 \end{cases}$$

<u>**Answers**</u> <u>for Pencil Problems</u> <u>*(Textbook Exercise references in parentheses)*</u>:

1. $z = 125x + 200y$ *(5.6 #15a)* **2a.** $x \leq 450$ *(5.6 #15b)* **2b.** $y \leq 200$ *(5.6 #15b)*

2c. $600x + 900y \leq 360,000$ *(5.6 #15b)* **2d.** $z = 125x + 200y$; $\begin{cases} x \leq 450 \\ y \leq 200 \\ 600x + 900y \leq 360,000 \end{cases}$ *(5.6 #15b)*

3a. 300 QLED and 200 OLED televisions; Maximum profit: \$77,500 *(5.6 #15e)* **3b.** 24 *(5.6 #7)*